ECE 521
HW # 3 SOLUTION

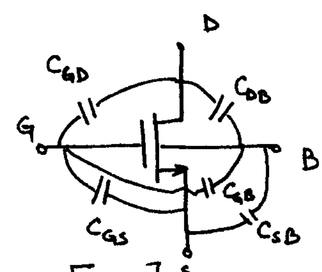
1. a) $I_D = f(V_{GS}, V_{DS}, V_{BS})$

$$\frac{\partial I_D}{\partial V_{GS}} = g_m, \qquad \frac{\partial I_D}{\partial V_{DS}} = g_{DS}, \qquad \frac{\partial I_D}{\partial V_{BS}} = g_{mb}$$

$$I_K = I_D^k - g_m V_{GS}^k - g_{DS} V_{DS}^k - g_{mb} V_{BS}^k$$

DC stamp

$$\begin{array}{c|c} & \begin{array}{cccc} D & G & S & B \end{array} \\ \begin{array}{c} D \\ G \\ S \\ B \end{array} & \begin{bmatrix} g_{DS} & g_m & -g_m - g_{ds} - g_{mb} & g_{mb} \\ 0 & 0 & 0 & 0 \\ -g_{ds} & -g_m & g_m + g_{ds} + g_{mb} & -g_{mb} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \text{RHS} \\ \begin{bmatrix} -I_K \\ 0 \\ I_K \\ 0 \end{bmatrix} \end{array}$$



b) AC stamp

$$\begin{array}{c|c} & \begin{array}{cccc} D & G & S & B \end{array} \\ \begin{array}{c} D \\ G \\ S \\ B \end{array} & \begin{bmatrix} g_{DS} + j\omega(C_{GD} + C_{DB}) & g_m - j\omega C_{GD} & -g_m - g_{DS} - g_{mb} & g_{mb} - j\omega C_{DB} \\ -j\omega C_{GD} & j\omega(C_{GD} + C_{GS} + C_{GB}) & -j\omega C_{GS} & -j\omega C_{GB} \\ -g_{DS} & -g_m - j\omega C_{GS} & (g_m + g_{DS} + g_{mb}) + j\omega(C_{GS} + C_{SB}) & -g_{mb} - j\omega C_{SB} \\ -j\omega C_{DB} & -j\omega C_{GB} & -j\omega C_{SB} & j\omega(C_{GB} + C_{DB} + C_{SB}) \end{bmatrix} \end{array} \qquad \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \text{RHS} \end{array}$$

2 a) BE : $x_n = x_{n-1} + h\dot{x}_n \Rightarrow \dot{x}_n = \frac{1}{h}x_n - \frac{1}{h}x_{n-1}$

$$\Rightarrow \alpha = \frac{1}{h}, \quad \beta = \frac{-1}{h}x_{n-1}$$

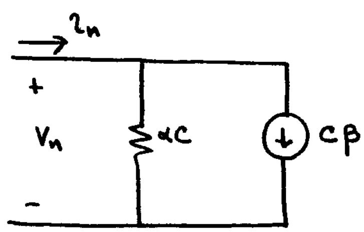
TR : $x_n = x_{n-1} + \frac{h}{2}(\dot{x}_n + \dot{x}_{n-1}) \Rightarrow \dot{x}_n = \frac{2}{h}x_n - \frac{2}{h}x_{n-1} - \dot{x}_{n-1}$

$$\Rightarrow \alpha = \frac{2}{h}, \quad \beta = \frac{-2}{h}x_{n-1} - \dot{x}_{n-1}$$

b) For capacitor $i = C\frac{dv}{dt}$

$$\Rightarrow i_n = C\left.\frac{dv}{dt}\right|_{t_n} = C(\alpha v_n + \beta) = \alpha C\, v_n + C\beta$$

$\Rightarrow$ companion model & stamp

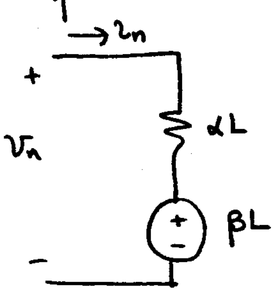


$$\begin{bmatrix} \alpha C & -\alpha C \\ -\alpha C & \alpha C \end{bmatrix} \quad \begin{bmatrix} -C\beta \\ C\beta \end{bmatrix}$$

RHS

For inductor $v_n = L\left.\frac{di}{dt}\right|_{t_n} = L(\alpha i_n + \beta) = \alpha L i_n + \beta L$

$\therefore$ companion model & stamp are

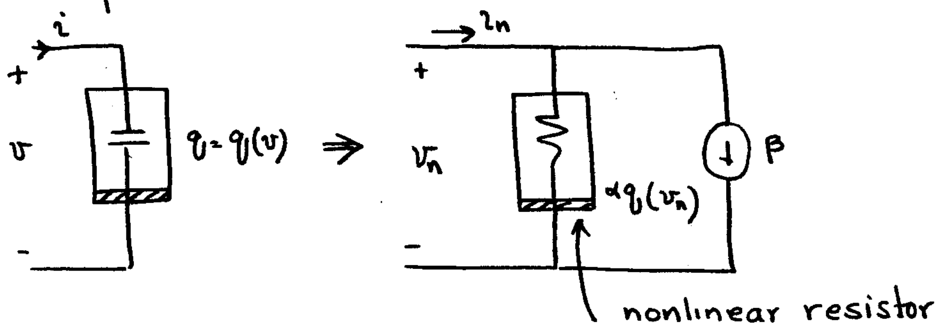


$$\begin{array}{c|ccc} & V_+ & V_- & i_L \\ \hline V_+ & & & 1 \\ V_- & & & -1 \\ BCR & 1 & -1 & -\alpha L \end{array} \qquad \text{RHS} \begin{bmatrix} 0 \\ 0 \\ \beta L \end{bmatrix}$$

c) $q = q(v)$

$$i = \frac{dq}{dt} \Rightarrow i_n = \left.\frac{dq}{dt}\right|_{t_n} = \alpha q_n + \beta = \alpha q(v_n) + \beta$$

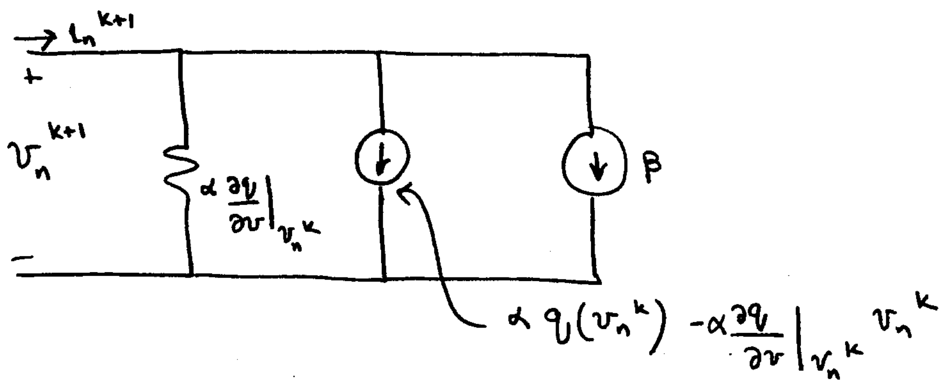
$\therefore$ Companion model at t_n



<u>Newton iteration</u>

$$q(v_n^{k+1}) = q(v_n^k) + \left.\frac{\partial q}{\partial v}\right|_{v_n^k} (v_n^{k+1} - v_n^k)$$

$\therefore$ companion model



<u>Note</u> Use charge as the quantity of interest not $\frac{\partial q}{\partial v}$!

Note Solutions to problem 3 courtesy Prof. Jacob White of MIT.

3 **(a)** The Local Error (LE) of a multistep method is defined to be the result of plugging the exact solution $x(t)$ into the multistep equation. In other words, the LE is a measure of how well the exact solution fits into the multistep method.

From class notes we have $\frac{x_n}{2} - \frac{x_{n-2}}{2} - h\dot{x}_{n-1} = 0$ is a second-order method with

$$\alpha_0 = 1/2, \quad \alpha_1 = 0, \quad \alpha_2 = -1/2, \quad \beta_0 = 0, \quad \beta_1 = -1, \quad \beta_2 = 0$$

(Note $p = 2$, $k = 2$ & $\sum_{i=0}^{p} \beta_i = -1$)

Since method is second order

$$\Rightarrow \text{LE} = c_3 h^3 \dddot{x}(t_n)$$

where $c_3 = \frac{1}{3!}\left(\sum_{i=0}^{2} \alpha_i (-i)^3 + 3\sum_{i=0}^{2} \beta_i (-i)^2\right)$

$$= \frac{1}{6}\left[\alpha_1(-1)^3 + \alpha_2(-2)^3 + 3\beta_1(-1)^2 + 3\beta_2(-2)^2\right]$$

$$= \frac{1}{6}\left[-\frac{1}{2}(-8) + 3(-1)\right] = \frac{1}{6}(4-3) = \frac{1}{6}$$

$$\therefore \text{LE} = \frac{h^3}{6}\dddot{x}(t_n)$$

(b)

Consider that the stability polynomial for the midpoint formula is

$$z^2 - 2h\lambda z - 1.$$

Its two roots z are

$$z_{1,2} = h\lambda \pm \sqrt{1 + (h\lambda)^2}$$

and for any real $h\lambda < 0$ one root is always outside the unit circle ($|z_1| > 1$).

No matter how small the timestep is, if $Re(\lambda) \neq 0$, $h\lambda$ will not lie in the region of absolute stability and that could lead you to think that the midpoint formula is not convergent. However, this is not the case. We saw in part (a) that the formula is consistent and clearly the roots as $h \to 0$ are

$$z_{1,2} = \pm 1$$

which are simple on the unit disk. Therefore, the midpoint formula is convergent, which means that if you shrink the timestep enough you will eventually converge.

(c,d)

The plots obtained with the midpoint formula using, respectively, $h = 2, h = 0.5$ and $h = 0.1$, applied to the given problem are shown in Figures 1, 2 and 3. As can be seen from the plots the method is unstable for finite h. As $h \to 0$, however, the instability weakens, as we expect, since the method is in fact stable as $h \to 0$, thus convergent.

Study this example until you understand the meaning of convergence, and the difference between small-h stability and time-stability!

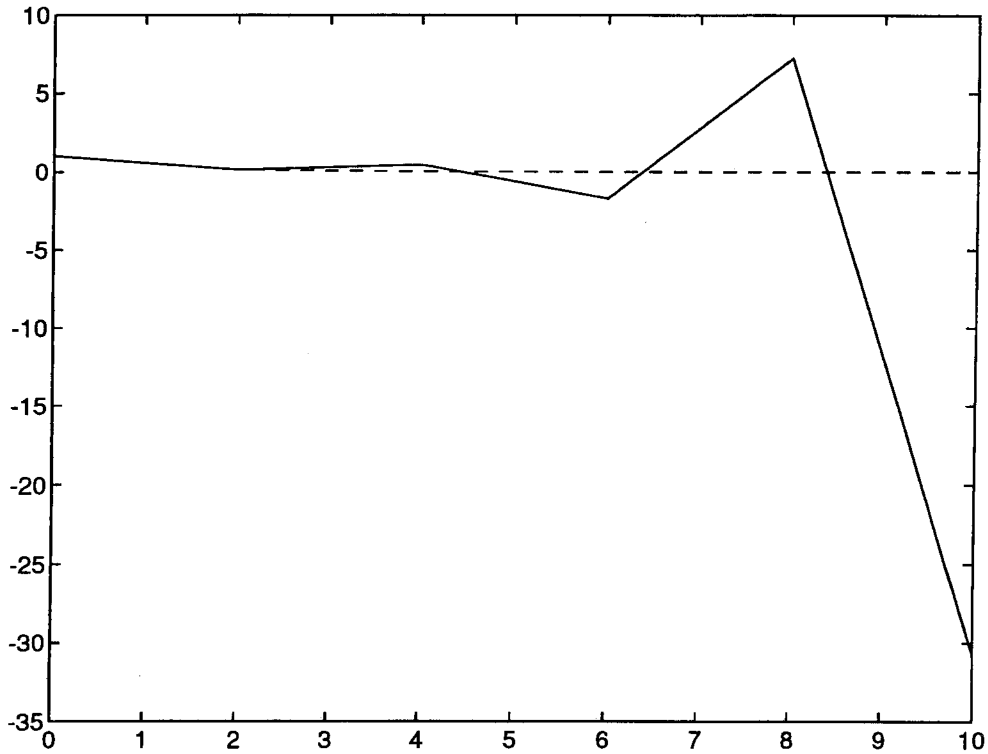


Figure 1: Solution obtained using the midpoint formula, for $h = 2$.

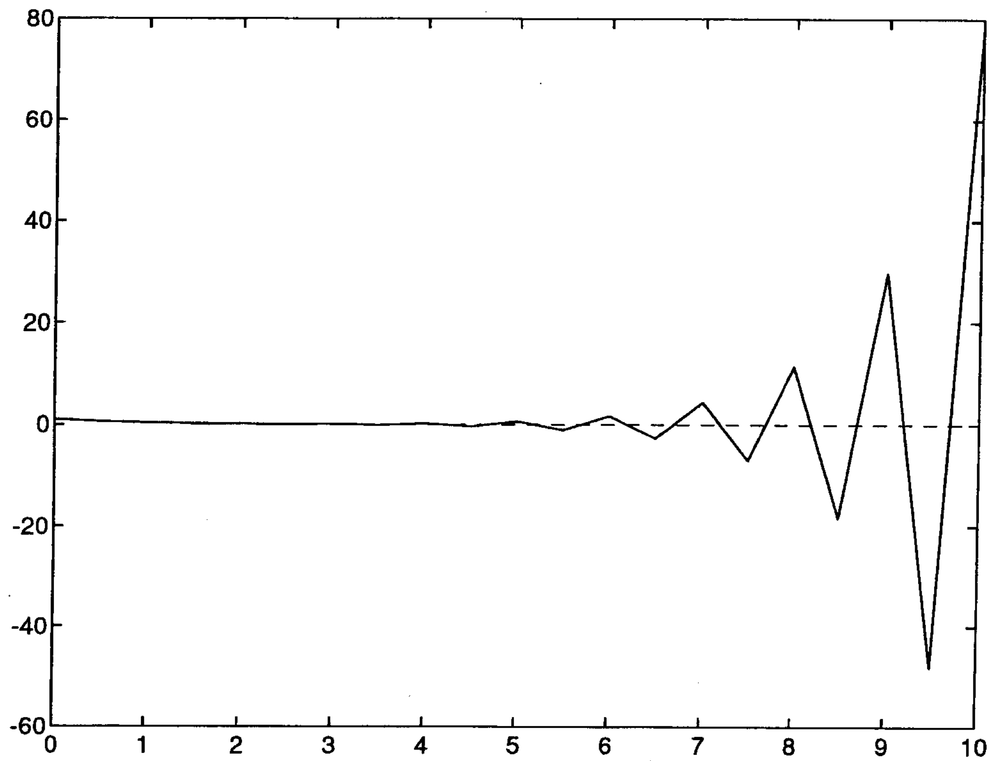


Figure 2: Solution obtained using the midpoint formula, for $h = 0.5$.

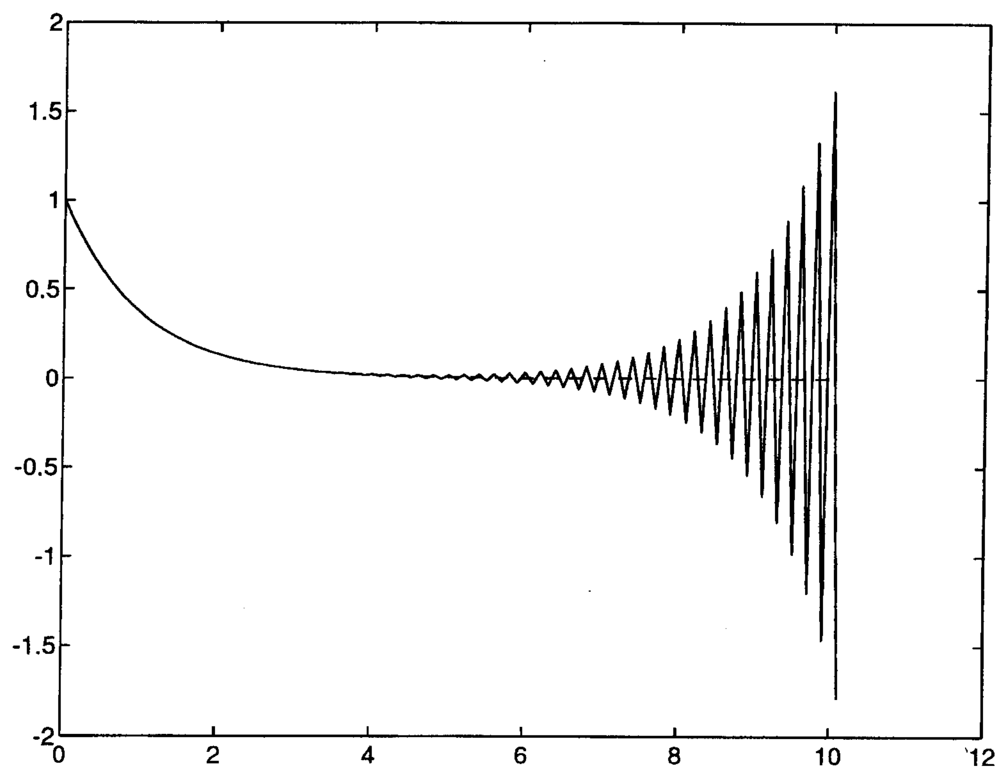


Figure 3: Solution obtained using the midpoint formula, for $h = 0.1$.

Solution from Ruoxin Jiang

4) (a) $x_n - \alpha_1 x_{n-1} - h\beta_0 \dot{x}_n - h\beta_1 \dot{x}_{n-2} = 0$

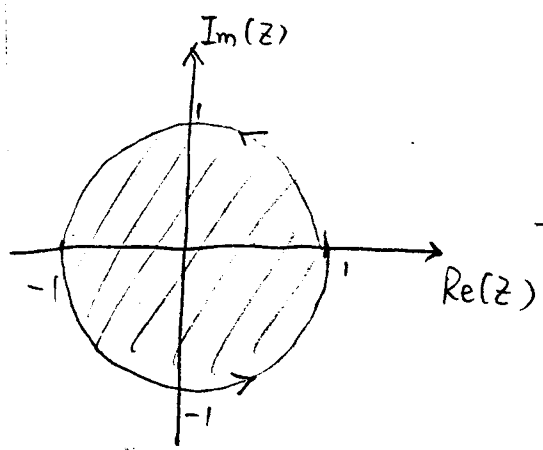
$$\left.\begin{array}{l} E^{(0)}[X,0]=0 \Rightarrow 1-\alpha_1 = 0 \\ E^{(1)}[X,0]=0 \Rightarrow \alpha_1 - \beta_0 - \beta_1 = 0 \\ E^{(2)}[X,0]=0 \Rightarrow -\alpha_1 + 4\beta_1 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \alpha_1 = 1 \\ \beta_0 = \frac{3}{4} \\ \beta_1 = \frac{1}{4} \end{array}\right.$$

(b) $$LE_n = \frac{E^{(3)}[X,0]}{3!} \cdot h^3 = -\frac{1}{3} h^3 x^{(3)}(t_n)$$

(c) $$x_n - x_{n-1} - \frac{3h}{4}\dot{x}_n - \frac{h}{4}\dot{x}_{n-2} = 0$$

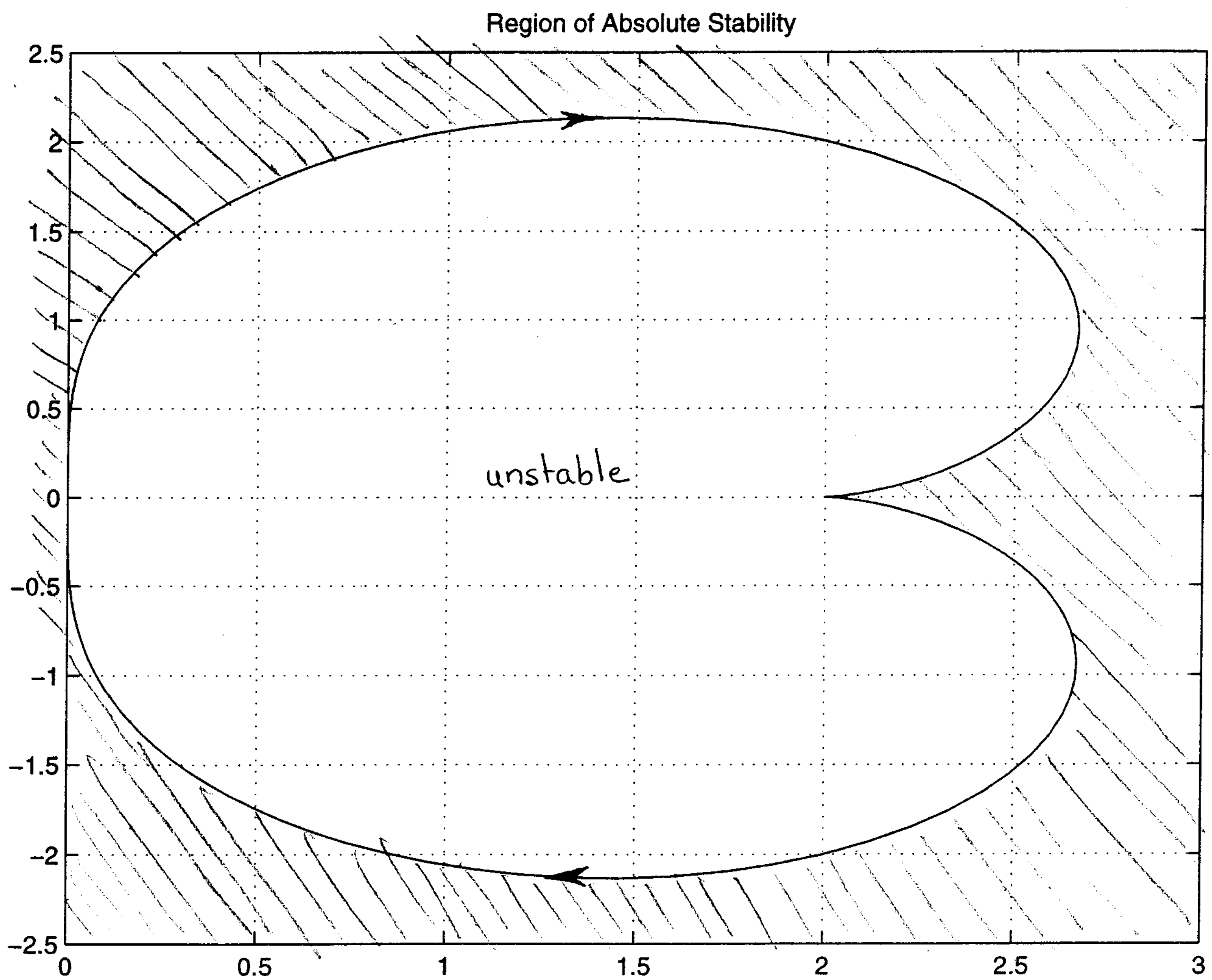
$$z^2 - z - \frac{3\lambda z^2}{4} - \frac{1}{4}\lambda = 0$$

$$\lambda = \frac{z^2 - z}{\frac{3z^2}{4} + \frac{1}{4}} = 4 \times \left[\frac{[\cos 2\theta - \cos\theta] + j[\sin 2\theta - \sin\theta]}{[3\cos 2\theta + 1] + j[3\sin 2\theta]}\right]$$



$\Rightarrow$ region of absolute stability, see next page.

problem # 4(c)


Region of Absolute Stability
unstable
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
0.5
1
1.5
2
2.5
3