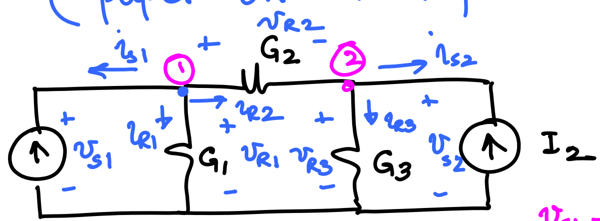
Office Hours: Tu/Th 5-6 pm KEC 4095
& by appointment

Formulation of circuit equations
- Nodal analysis (NA)
- Modified nodal analysis (MNA)
(paper on MNA posted on class website)

$i_{s1} = -I_1$

$i_{R1} - G_1 v_{R1} = 0$

$i_{R2} - G_2 v_{R2} = 0$



V_1, V_2 are the node voltages

$v_{s1} = v_1$

$v_{R1} = v_1$

$v_{R2} = v_1 - v_2$

$v_{R3} = v_2$

$v_{s2} = v_2$

KCL at nodes 1 & 2

$$i_{s1} + i_{R1} + i_{R2} = 0$$

$$-i_{R2} + i_{R3} + i_{s2} = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}}_{\substack{A \\ \text{incidence matrix}}} \begin{bmatrix} i_{s1} \\ i_{R1} \\ i_{R2} \\ i_{R3} \\ i_{s2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

General form $A I_b = 0$ KCL

Branch to node voltage relations

$$\begin{bmatrix} v_{s1} \\ v_{R1} \\ v_{R2} \\ v_{R3} \\ v_{s2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$V_b = A^T V_n$$

BCR or BCE

$$\begin{bmatrix} i_{s1} \\ i_{R1} \\ i_{R2} \\ i_{R3} \\ i_{s2} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & G_1 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 \\ 0 & 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{s1} \\ v_{R1} \\ v_{R2} \\ v_{R3} \\ v_{s2} \end{bmatrix} = \begin{bmatrix} -I_1 \\ 0 \\ 0 \\ 0 \\ -I_2 \end{bmatrix}$$

$$I_b - G_b V_b = S$$

In general: $K_i I_b + K_v V_b = S$

(Augmented) Incidence Matrix A'

(n+1) nodes

b branches

$$
A' = \begin{array}{c|c}
 & \xrightarrow{\text{branches}} \\
 & 1\ 2\ 3\ \cdots\ j\ \cdots\ b \\
\begin{array}{c} \text{nodes} \downarrow \\ 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ \text{includes ground} \rightarrow n+1 \end{array} & \left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ - & - & - & (+1,-1,0) & \\ & & & & \\ \end{array} \right]
\end{array}
$$

$A_{ij} = +1$ if node i is the '+' terminal of branch j

$A_{ij} = -1$ if node i is the '−' terminal branch j

$A_{ij} = 0$ if node i is not connected to branch j

All columns will have 2 entries & sum up to zero

The Sparse Tableau Analysis (STA) 1969-71 by IBM research

KCL $\quad A I_b = 0$

$$
n \text{ nodes} \updownarrow \left[\quad A \quad \right] \begin{bmatrix} I_1 \\ \vdots \\ I_b \end{bmatrix} = 0
$$

KVL $\quad V_b = A^T V_n \Rightarrow V_b - A^T V_n = 0$

BCR $\quad K_i I_b + K_v V_b = S$

Ex

R: $\quad I_b - G V_b = 0$

V: $\quad 0 I_b - V_b = V_s$

$A I_b = 0$

$V_b - A^T V_n = 0$

$K_i I_b + K_v V_b = S$

$$
\begin{bmatrix} I_b \\ \hline V_b \\ \hline V_n \end{bmatrix}
$$

$$\begin{array}{cc} & \begin{array}{ccc} \;\; b \;\; & \;\; b \;\; & \;\; n \;\; \end{array} \\ \begin{array}{c} n \\ b \\ b \end{array} & \left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & I & -A^T \\ \hline K_i & K_v & 0 \end{array}\right] \end{array} \begin{bmatrix} I_{b1} \\ I_{b2} \\ \vdots \\ I_{bb} \\ \hline V_{b1} \\ \vdots \\ V_{bb} \\ \hline V_{n1} \\ \vdots \\ V_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 0 \\ \hline S \end{bmatrix}$$

$n+2b$ unknowns; $n+2b$ equations

A has at most $2b$ non zero entries

A^T " " " $2b$ " "

I has b nonzeros

K_i, K_v have at most b non zero entries

Maximum # of non zero entries $2b + 2b + b + b + b = 7b$

Matrix size $(n+2b) \times (n+2b)$ $\Rightarrow$ very sparse matrix

Summary of STA:

- can be applied to any circuit
- Equations can be assembled by inspection
- STA matrix is very sparse

Derive NA from STA

$$A I_b = 0$$

$$V_b - A^T V_n = 0$$

$$K_i I_b + K_v V_b = S$$

For NA
$$I_b = \underbrace{K_i^{-1}}_{K_i \text{ is invertible}} [S - K_v V_b]$$

$$A I_b = A K_i^{-1} [S - K_v \underbrace{V_b}_{A^T V_n}] = 0$$

$$A K_i^{-1} [S - K_v A^T V_n] = 0$$

$$\underbrace{A k_i^{-1} k_v A^T}_{\text{NA matrix}} \underbrace{V_n}_{\text{node voltages}} = \underbrace{A k_i^{-1} S}_{\text{NA source}}$$

General Form

$Ax = b$ is a system of equations

A ← not the incidence matrix

$x = A^{-1} b$ — A is a nonsingular matrix

Solution methods

1) Direct methods — Gaussian Elimination, LU Decomposition

2) Indirect methods (iterative methods)
   - Gauss Jacobi/Seidel
   - ⋮

Gaussian Elimination (GE)

$$Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\text{row } 2 = \text{row } 2 - \frac{a_{21}}{a_{11}} \text{row } 1$$

multiplies

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \boxed{\frac{a_{21}}{a_{11}}} a_{12} & \cdots & \\ 0 & & & \\ \vdots & & & \\ 0 & a_{n2} - \boxed{\frac{a_{n1}}{a_{11}}} a_{12} & \cdots & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \frac{a_{21}}{a_{11}} b_1 \\ \vdots \\ b_n - \frac{a_{n1}}{a_{11}} b_1 \end{bmatrix}$$

$a_{22}^{(2)}$ → $a_{22} - \frac{a_{21}}{a_{11}} a_{12}$

$a_{n2}^{(2)}$ → $a_{n2} - \frac{a_{n1}}{a_{11}} a_{12}$

$b_2^{(2)}$ ⋮ $b_n^{(2)}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & & & \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & \ddots & \vdots \\ & & & a_{nn}^{(n)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(2)} \\ \vdots \\ b_n^{(n)} \end{bmatrix}$$

This is an upper triangular system

Solve for $x_n = \dfrac{b_n^{(n)}}{a_{nn}^{(n)}}$

$$x_{n-1} = \frac{b_{n-1}^{(n-1)} - a_{n-1,n}^{(n-1)} x_n}{a_{n-1,n-1}^{(n-1)}}$$

$$\vdots$$

$$x_1 = \frac{b_1 - \sum_{j=2}^{n} a_{1j} x_j}{a_{11}}$$

This process is called backward substitution

Overall GE: 1) triangularization
$A \longrightarrow$ Upper triangular matrix U

2) backward substitution/solve

Triangularization (assume $A[i][i] \neq 0$)

For $i = 1$ to n { each row $\alpha = 1/A[i][i]$

1 division

For $j = i+1$ to n { each row below row i

$m = \alpha * A[j][i]$

$(n-i)$ mult

For $k = i+1$ to n { traverse row j

$A[j][k] = A[j][k] - m * A[i][k]$

$(n-i)(n-i)$ mult

$(n-i)(n-i)$ addition

}

(n-i) mult $\quad b[j] = b[j] - m*b[i]$

}

}

Total # of mult/div $\quad 1 + \underbrace{(n-i) + (n-i)(n-i)}_{(n-i)(n-i+1)} + \underbrace{(n-i)}_{\text{RHS}}$

$$\sum_{i=1}^{n} \underbrace{(n-i)(n-i+1)}_{\frac{n^3-n}{3}} + \underbrace{(n-i)}_{\frac{n^2-n}{2}} + 1 \qquad \sim \frac{n^3}{3}$$

Backward substitution $\sim n^2/2$

What about solving
$$\begin{aligned} Ax &= b_A \\ &= b_B \\ &\vdots \\ &= b_X \end{aligned}$$

$$A \rightarrow \begin{bmatrix} \diagdown & U \\ 0 & \diagdown \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \text{RHS}$$

LU Factorization (or Decomposition)

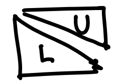
$$A \rightarrow \begin{bmatrix} \diagdown & U \\ L & \diagdown \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = b$$

$$\begin{aligned} b_1^{(1)} &= b_1 \\ b_2^{(2)} &= b_2 - \overbrace{\frac{a_{21}}{a_{11}}}^{l_{21}} b_1^{(1)} \\ &\vdots \\ b_n^{(n)} &= b_n - \underbrace{\frac{a_{n1}}{a_{11}}}_{l_{n1}} b_1^{(1)} - \underbrace{\frac{a_{n2}}{a_{22}}}_{l_{n2}} b_2^{(2)} \cdots \overset{l_{n-1,n-1}}{\cdot} \end{aligned}$$

$$\begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & & & 1 & \\ l_{n1} & l_{n2} & \cdots & l_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ \vdots \\ b_n^{(n)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$Ax = b$$

$$L\underbrace{Ux}_{y} = b$$



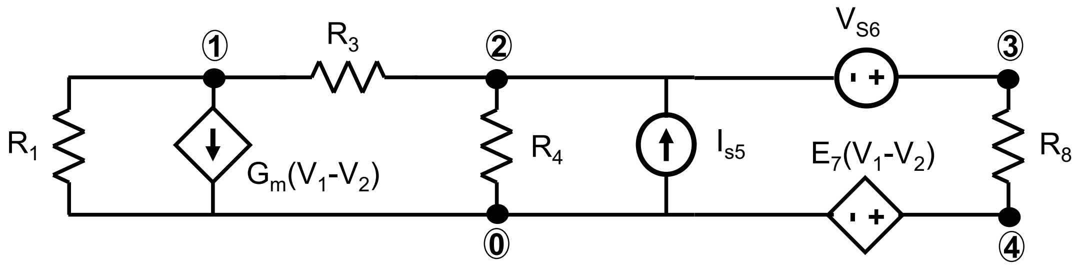
$\sim n^2/2$ $\quad Ly = b.$ $\quad$ Forward substitution

$\sim n^2/2$ $\quad Ux = y$ $\quad$ Backward substitution

One time operation LU Factorization $\sim n^3/3$

Repeated Forward/Back solves for different RHS $\sim n^2$

MNA Example



$$\left[\begin{array}{cccc|cc} \frac{1}{R_1}+G_m+\frac{1}{R_3} & -\left(G_m+\frac{1}{R_3}\right) & 0 & 0 & 0 & 0 \\ -\frac{1}{R_3} & \frac{1}{R_3}+\frac{1}{R_4} & 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{R_8} & -\frac{1}{R_8} & 1 & 0 \\ 0 & 0 & -\frac{1}{R_8} & \frac{1}{R_8} & 0 & 1 \\ \hline 0 & -1 & 1 & 0 & 0 & 0 \\ E_7 & -E_7 & 0 & -1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hline i_6 \\ i_7 \end{array}\right] = \left[\begin{array}{c} 0 \\ I_{s5} \\ 0 \\ 0 \\ \hline V_{S6} \\ 0 \end{array}\right]$$

$$\left[\begin{array}{c|c} Y_n & B \\ \hline C & 0 \end{array}\right] \left[\begin{array}{c} v \\ \hline i \end{array}\right] = S$$

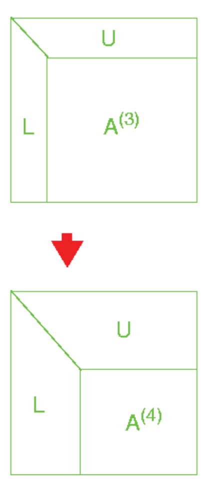
From: A. Nardi

Pivoting for Accuracy

Example 1: After two steps of G.E. MNA matrix becomes:

$$\begin{bmatrix} x & x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & x & 0 \\ 0 & 0 & \frac{1}{R} & -\frac{1}{R} & 1 & 0 \\ 0 & 0 & -\frac{1}{R} & \frac{1}{R} & 0 & 1 \\ 0 & 0 & x & 0 & x & 0 \\ 0 & 0 & 0 & x & x & 0 \end{bmatrix}$$

$$\begin{bmatrix} x & x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & x & 0 \\ 0 & 0 & \frac{1}{R} & -\frac{1}{R} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & x & x & 0 \end{bmatrix}$$



$$l_{i4} = \frac{a_{i4}^{(4)}}{a_{44}^{(4)}} = \infty \ !!!$$

0

From: A. Sangiovanni-Vincentelli

Solution:

Interchange rows and/or columns to bring non-zero element into position (k,k):

$$\begin{bmatrix} 0 & 1 & 1 \\ x & x & 0 \\ x & x & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x & x & 0 \\ 0 & 1 & 1 \\ x & x & 0 \end{bmatrix}$$

Example 2:

$$\begin{bmatrix} 1.25 \times 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

solution to 5 digit accuracy

$x_1 = 1.0001$

$x_2 = 5.0000$

From: A. Sangiovanni-Vincentelli

Problem with Small Pivots

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

GE

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 0 & 12.5-1.25\cdot 10^{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75-6.25\cdot 10^{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5\,digits} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

From: A. Nardi

Problem with Small Pivots

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

GE

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 0 & 12.5 - 1.25\cdot 10^{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 - 6.25\cdot 10^{5} \end{bmatrix}$$

Rounded to 3 digits

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 0 & -1.25\cdot 10^{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ -6.25\cdot 10^{5} \end{bmatrix}$$

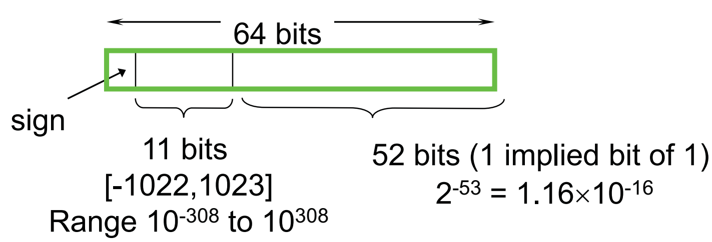
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3\,digits} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5\,digits} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

From: A. Nardi

Floating Point Arithmetic

Double precision number (IEEE Representation)



Machine precision: $\varepsilon \cong 10^{-16}$

$\Rightarrow$ $1 + \varepsilon = 1$

Avoid sum/subtraction of large and tiny numbers

$\Rightarrow$ Avoid big multipliers

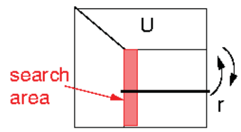
$\Rightarrow$ Do NOT use equalities in floating point arithmetic

Pivoting Strategies

1. Partial Pivoting: (row interchange only)

choose r as the smallest integer such that

$$\left|a_{rk}^{(k)}\right| = \max_{j=k,\ldots,n} \left|a_{jk}^{(k)}\right|$$

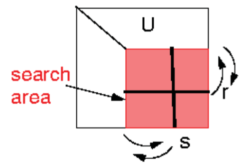


2. Complete Pivoting

(row and column interchange)

Choose r and s as the smallest integers such that

$$\left|a_{rk}^{(k)}\right| = \max_{\substack{i=k,\ldots,n \\ j=k,\ldots,n}} \left|a_{ij}^{(k)}\right|$$



3. Threshold Pivoting

a. Apply partial pivoting only if $\left|a_{kk}^{(k)}\right| < \varepsilon_p \left|a_{rk}^{(k)}\right|$

b. Apply complete pivoting only if $\left|a_{kk}^{(k)}\right| < \varepsilon_p \left|a_{rs}^{(k)}\right|$

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From: A. Sangiovanni-Vincentelli

Partial Pivoting for Small Pivots

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

swap

$$\begin{bmatrix} 12.5 & 12.5 \\ 1.25\cdot 10^{-4} & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

GE

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 - 12.5\cdot 10^{-5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 - 75\cdot 10^{-5} \end{bmatrix}$$

Rounded to 3 digits

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3\,digits} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5\,digits} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

From: A. Nardi