ECE 521 Lecture #9 Oct. 19, 2016

Note Title 10/19/2016

NumEqns = NumNodes + NumBranches + 1;

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Newton Loop numIter = 1; oldSolution [ ] = 0;

for (numIter ≤ MAXITERATIONS) {

sp Clear ( ); Clear RHS

ckt Load — resLoad, VsrcLoad

$$J(x^k)\underbrace{(x^{k+1}-x^k)}_{\Delta x^{k+1}} = -f(x^k)$$

spFactor( )

spSolve ( ) Solution

/*check for convergence */

RELTOL = 10^{-3}

VNTOL = 10^{-6}

$$|\Delta x| \le VNTOL + RELTOL * |x^k, x^{k+1}|$$

MAX

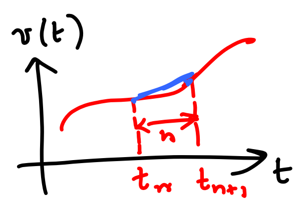
not converged

old Solution ← Solution

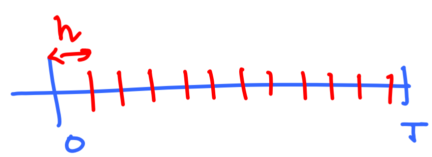
}

Companion Networks → Linear elements

Transient Analysis



$C\dfrac{dv}{dt}$; $L\dfrac{dI}{dt}$



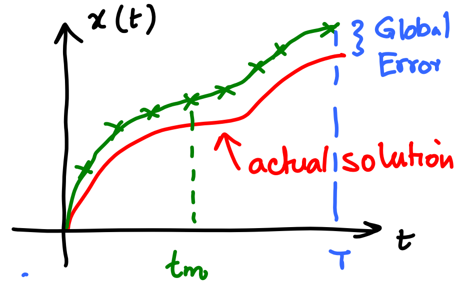
BE $$\left.\frac{dx}{dt}\right|_{t_{n+1}} = \frac{x(t_{n+1}) - x(t_n)}{\underbrace{t_{n+1} - t_n}_{h}}$$

FE $$\left.\frac{dx}{dt}\right|_{t_n} = \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}$$

Initial value problem

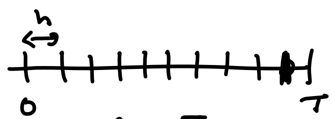
$$\dot{x} = \frac{dx}{dt} = f(x) \qquad x(0) = x_0$$

$$x(t) = \int f(x)\,dt$$



$\hat{x}(t_m)$ = computed solution

$x(t_m)$ = true solution

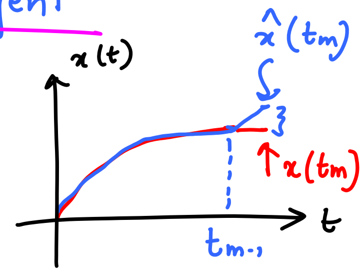


$$M = \frac{T}{h}$$

If $\displaystyle\lim_{h \to 0} \max_{0 \le m \le M} \left\| \hat{x}(t_m) - x(t_m) \right\| = 0$

then the method is convergent

Focus on one time step error



Given $\hat{x}(t_{m-1}) = x(t_{m-1})$

$$e_L(h) = \left| \hat{x}(t_m) - x(t_m) \right|$$

local error If $\displaystyle\lim_{h \to 0} \frac{e_L(h)}{h} = 0$ then consistent

Global error $\displaystyle\approx \sum_{i=0}^{M} e_{Li}$ where $M = \frac{T}{h}$

$$\le M \max e_L(h) = \frac{T}{h} \max e_L(h)$$

$$\lim_{h \to 0} \frac{T}{h} \max e_L(h)$$

Stability The single step errors don't grow too fast

Consistency + Stability $\Longleftrightarrow$ Convergent

BE:
$$\left.\frac{d}{dt} x(t)\right|_{(m+1)h} = \frac{x((m+1)h) - x(mh)}{h} \qquad \frac{dx}{dt} = f(x)$$

$$f(x(m+1)h)$$

$$x((m+1)h) = x(mh) + h f(x(\overline{m+1})h)$$

Implicit method

$$x((m+1)h) = x(mh) + h\, f(mh)$$

Explicit method

F.E. $$\left.\frac{dx}{dt}\right|_{mh} = \frac{x((m+1)h) - x(mh)}{h} = f(x(mh))$$

Consider $x(0)$ & $x(h)$

Computed solution $\hat{x}$; actual solution x

① $$\hat{x}(h) = x(0) + h\, f(\hat{x}(h))$$

$x(h), x(0)$

$$x(0) = x(h) + (-h)\dot{x}(h) + \frac{(-h)^2}{2}\ddot{x}(\tau)$$

$\tau \in [0,h]$

② $$x(h) = x(0) + h\underbrace{\dot{x}(h)}_{f(x(h))} - \frac{h^2}{2}\ddot{x}(\tau)$$

Subtract ② from ①: $$\hat{x}(h) - x(h) = h\left[f(\hat{x}(h)) - f(x(h))\right] + \frac{h^2}{2}\ddot{x}(\tau)$$

$$\|\hat{x}(h) - x(h)\| \le h\,\|f(\hat{x}(h)) - f(x(h))\| + \frac{h^2}{2}\|\ddot{x}(\tau)\|$$

If $f(x)$ is Lipschitz continuous

$$\|f(x) - f(y)\| \le L\,\|x - y\|$$

$$\|\hat{x}(h) - x(h)\| \le hL\,\|\hat{x}(h) - x(h)\| + \frac{h^2}{2}\|\ddot{x}(\tau)\|$$

$$(1 - hL)\,\|\hat{x}(h) - x(h)\| \le \frac{h^2}{2}\|\ddot{x}(\tau)\|$$

$$\|\hat{x}(h) - x(h)\| \le \frac{h^2}{2(1-hL)}\|\ddot{x}(\tau)\|$$

$$\lim_{h\to 0} \frac{\|\hat{x}(h) - x(h)\|}{h} \le \lim_{h\to 0} \frac{h}{2(1-hL)}\|\ddot{x}(\tau)\|$$

i.e. a consistent method

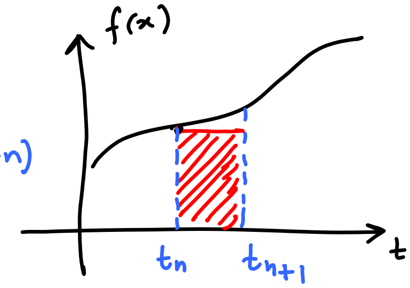
Summary

$$\dot{x} = f(x)$$

FE

$$\dot{x}_n = \frac{x_{n+1} - x_n}{h} = f(x_n)$$

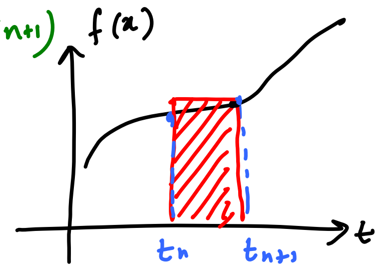
$$x_{n+1} = x_n + h f(x_n)$$



BE

$$\dot{x}_{n+1} = \frac{x_{n+1} - x_n}{h} = f(x_{n+1})$$

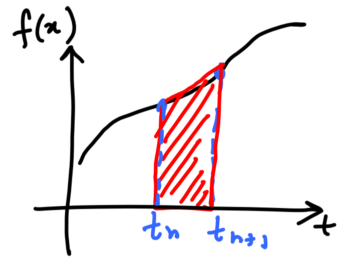
$$x_{n+1} = x_n + h f(x_{n+1})$$



TR (trapezoidal rule)

$$\frac{\dot{x}_{n+1} + \dot{x}_n}{2} = \frac{x_{n+1} - x_n}{h}$$

$$x_{n+1} = x_n + \frac{h}{2}\left[f(x_{n+1}) + f(x_n)\right]$$



- FE, BE, and TR are all one-step methods
  i.e., they use information from only the previous time point
- FE is an explicit method
- BE, TR are implicit methods

Linear multi step methods

$$\dot{x} = f(x) \qquad x(0) = x_0$$

$$\sum_{i=0}^{p} \alpha_i x_{n-i} + h \sum_{i=0}^{p} \beta_i \dot{x}_{n-i} = 0$$

is a p-step method

For $p = 1$:

$$\alpha_0 x_n + \alpha_1 x_{n-1} + h \beta_0 \dot{x}_n + h \beta_1 \dot{x}_{n-1} = 0$$

FE:

$$h \dot{x}_{n-1} = x_n - x_{n-1}$$

$$x_n - x_{n-1} - h \dot{x}_{n-1} = 0$$

$\alpha_0 = 1, \quad \alpha_1 = -1, \quad \beta_0 = 0, \quad \beta_1 = -1$

BE

$$h\dot{x}_n = x_n - x_{n-1}$$

$$x_n - x_{n-1} - h\dot{x}_n = 0$$

$$\alpha_0 = 1, \quad \alpha_1 = -1, \quad \beta_0 = -1, \quad \beta_1 = 0$$

TR

$$\frac{h}{2}(\dot{x}_n + \dot{x}_{n-1}) = x_n - x_{n-1}$$

$$x_n - x_{n-1} - \frac{h}{2}\dot{x}_n - \frac{h}{2}\dot{x}_{n-1} = 0$$

$$\alpha_0 = 1, \quad \alpha_1 = -1, \quad \beta_0 = -\frac{1}{2}, \quad \beta_1 = -\frac{1}{2}$$

For an explicit method $\beta_0 = 0$;

For an implicit method $\beta_0 \neq 0$

Linear multi step methods

p-step method

$$\sum_{i=0}^{p} \alpha_i x_{n-i} + h \sum_{i=0}^{p} \beta_i \dot{x}_{n-i} = 0$$

Local truncation Error (LTE)

$$LTE_n = \underset{\text{actual}}{x(t_n)} - \underset{\text{computed}}{x_n}$$

assuming no round-off error and that no previous error has been made

$$x_i = x(t_i) \qquad i = 1, 2, \ldots n-1$$

This is a one-step error assuming past data is exact

Local Error (LE)

$$\sum_{i=0}^{p} \alpha_i x_{n-i} + h \sum_{i=0}^{p} \beta_i \dot{x}_{n-i} = 0$$

computed values

$$\rightarrow LE_n = \sum_{i=0}^{p} \alpha_i x(t_{n-i}) + h \sum_{i=0}^{p} \beta_i \dot{x}(t_{n-i})$$

it is the amount by which the exact solution fails to satisfy the LMS formula

$$LTE_n = x(t_n) - x_n$$

↑ all past data is exact

Assume $\alpha_0 = 1$

$$x_n + \sum_{i=1}^{n} \alpha_i x(t_{n-i}) + h\beta_0 \dot{x}_n + h\sum_{i=1}^{n} \beta_i \dot{x}(t_{n-i}) = 0$$

$x(t_n) - x(t_n)$ $\qquad$ $h\beta_0 \dot{x}(t_n) - h\beta_0 \dot{x}(t_n)$

$$x_n + \sum_{i=0}^{n} \alpha_i x(t_{n-i}) - x(t_n) + h\sum_{i=0}^{n} \beta_i \dot{x}(t_{n-i}) + h\beta_0\left(\dot{x}_n - \dot{x}(t_n)\right) = 0$$

$$LTE_n = x(t_n) - x_n = \overbrace{\boxed{\sum_{i=0}^{n} \alpha_i x(t_{n-i}) + h\sum_{i=0}^{n} \beta_i \dot{x}(t_{n-i})}}^{= LE_n} + h\beta_0\left(f(x_n) - f(x(t_n))\right)$$

$$|LTE_n| \le |LE_n| + h\beta_0 \underbrace{|f(x_n) - f(x(t_n))|}_{L|x_n - x(t_n)|}$$

$$(1 - h\beta_0 L)|LTE_n| \le |LE_n|$$

$$|LTE_n| \le \frac{|LE_n|}{1 - h\beta_0 L}$$

- If $\beta_0 = 0$ $\quad LTE_n = LE_n$ i.e. for an explicit method

- LE bounds the LTE and we will now focus on LE.

$$LE = \sum_{i=0}^{p} \alpha_i x(t_{n-i}) + h \sum_{i=0}^{p} \beta_i \dot{x}(t_{n-i})$$

$$E[x(t), h] = \sum_{i=0}^{p} \alpha_i x(t_n - ih) + h \sum_{i=0}^{p} \beta_i \dot{x}(t_n - ih)$$

$$= E[x, 0] + h E^{(1)}[x, 0] + \frac{h^2}{2} E^{(2)}[x, 0], \dots + \frac{h^{k+1}}{(k+1)!} E^{(k+1)}[x, 0] + O(h^{k+2})$$

$$E^{(1)}[x, h] = \sum_{i=0}^{p} \alpha_i \dot{x}(t_n - ih)(-i) + \sum_{i=0}^{p} \beta_i \dot{x}(t_n - ih) + h \sum_{i=0}^{p} \beta_i \ddot{x}(t_n - ih)(-i)$$

$$E^{(1)}[x, 0] = \sum_{i=0}^{p} \alpha_i \dot{x}(t_n)(-i) + \sum_{i=0}^{p} \beta_i \dot{x}(t_n)$$

Definition

A multistep method is said to be a kth-order method if it is exact for polynomials of degree $\leq k$

$$a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k$$

$$E^{(i)}[x, 0] = 0 \qquad \text{for } 0 \leq i \leq k$$

Exactness constraints