ECE 521 Lecture # 11 Oct. 26, 2016

Note Title 10/26/2016

HW # 2 Due Today

HW # 3 handed out

Part 1 Due Nov. 14 (last part 1)

Part 2 Due Nov. 7

Linear multistep methods (integration methods)

p = # of steps $\sum_{i=0}^{p} \alpha_i x_{n-i} + h \beta_i \dot{x}_{n-i} = 0$

$\beta_0 \neq 0$ for an implicit method

k = order of integration method

$$\sum_{i=0}^{p} \alpha_i = 0$$

$$\sum_{i=0}^{p} \alpha_i (-i) + \beta_i = 0$$

$$\vdots$$

$$\sum_{i=0}^{p} \alpha_i (-i)^k + k \beta_i (-i)^{k-1} = 0$$

Exactness Constraints

$$LE = \frac{1}{(k+1)!} \left[\sum_{i=0}^{p} \alpha_i (-i)^{k+1} + (k+1) \beta_i (-i)^k \right] h^{k+1} x^{(k+1)}(t_n)$$

for BE: $LE = -\frac{1}{2} h^2 \ddot{x}(t_n)$

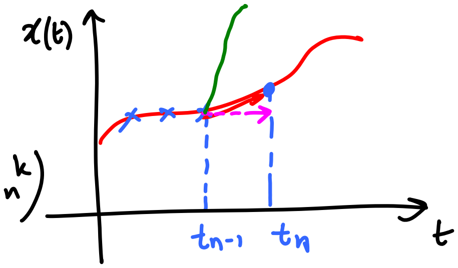
for TR: $LE = -\frac{1}{12} h^3 \dddot{x}(t_n)$

We want to solve $\dot{x} = f(x)$

$$\underline{\alpha_0 x_n} + \sum_{i=1}^{p} \alpha_i x_{n-i} + h \beta_0 \underbrace{\dot{x}_n}_{h\beta_0 f(x_n)} + h \sum_{i=1}^{p} \beta_i \dot{x}_{n-i} = 0$$

$$F(x_n) = \alpha_0 x_n + h \beta_0 f(x_n) + \text{constant} = 0$$

To determine x_n Newton's method is used:



$$\left.\frac{\partial F}{\partial x_n}\right|_{x_n^k} \left(x_n^{k+1} - x_n^k\right) = -F(x_n^k)$$

$x_n^0 = x_{n-1}$ is the initial guess for Newton method

For h small enough x_{n-1} is a good initial guess for Newton's method

Could do better by predicting the solution x_n^0

For a first-order method (BE)

$$x_n^0 = x_{n-1} + h\,\dot{x}_{n-1} \qquad \text{(FE)}$$

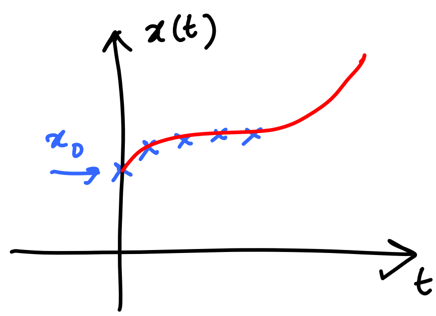
→ Predictor-corrector methods

For a p-step method:

What is x_0?

DC operating point solution

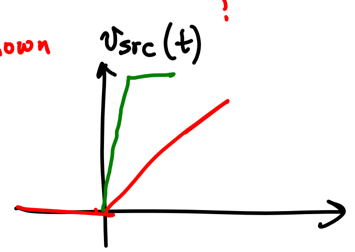
$$x_1 = x_0 + h\dot{x}_1 \qquad \text{(BE)}$$



So transient analysis typically starts with the BE method.

What about TR? $x_1 = \underset{\uparrow\ \text{Known}}{x_0} + \frac{h}{2}\left(\dot{x}_1 + \underset{?}{\dot{x}_0}\right)$

Convergence issues because of inconsistent initial conditions

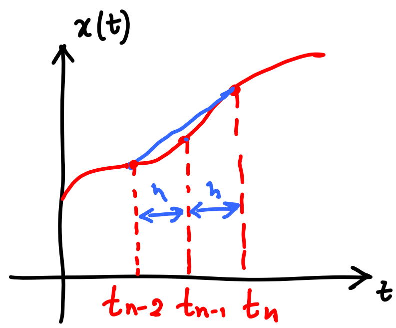


For a Kth order integration method $(k \geq 1)$

$$\lim_{h \to 0} \frac{LTE}{h} = 0 \qquad LE \propto h^{k+1}$$

$\Rightarrow$ the method is consistent

How do local errors accumulate? $\rightarrow$ Stability of integration methods



Example   mid-point method

$$\dot{x}_{n-1} = \frac{x_n - x_{n-2}}{2h}$$

$$x_n - x_{n-2} - 2h\dot{x}_{n-1} = 0$$

$\alpha_0 = 1, \ \alpha_1 = 0, \ \alpha_2 = -1, \ \beta_0 = 0, \ \beta_1 = -2, \ \beta_2 = 0$

2-step method (Explicit method)

This is a second order method (work thru exactness const)

$\Rightarrow$ Consistent method (local behavior OK!)

Consider a test problem $\dot{x} = -x, \ x(0) = 1$

Solution is $x(t) = e^{-t}$

$$x_n = x_{n-2} + 2h\dot{x}_{n-1}$$

$$x_2 = x_0 + 2h\dot{x}_1$$

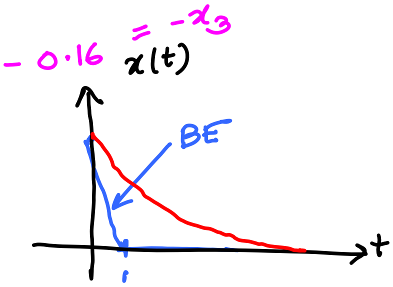
Take $h = 1$; from the exact solution $x_1 = e^{-1} = -\dot{x}_1$
$= 0.368$

$$x_2 = 1 + 2 \times 1(-0.368) = 0.264$$

$$\dot{x}_2 = -0.264$$

$$x_3 = x_1 + 2h\dot{x}_2 = -0.16 = -\dot{x}_3$$

Observe that $x_3 < 0$ which is incorrect!



$x_4 = x_2 + 2h\,\dot{x}_3 = 0.584$

$x_5 = -1.328$

$x_6 = 3.24$

$x_7 = -7.808$

Solution blows up

Try $h = 0.1$ ... $x_{9.9} = 44.03$, $x_{10} = -48.65$

$h = 0.01$ ... $x_{12} = 12124.17$

No matter how small h is the solution blows up

What about FE: $x_n = x_{n-1} + h\dot{x}_{n-1}$

$x_1 = x_0 + h\dot{x}_0$

$= 1 + 1(-1) = 0$

$\dot{x}_1 = 0$

$x_2 = 0$, $x_3 = 0$, 0 . .

Take $h = 3$: $x_n = x_{n-1} + 3\dot{x}_{n-1}$

$x_1 = 1 + 3(-1) = -2$

$x_2 = 4$

$x_3 = -8$

$x_4 = 16$

the solution starts growing again!

BE & TR will not have solutions that grow!

Three types of methods:

1) "Good" methods – stable regardless of value of h (BE, TR)

2) semi-good " – stable for some values of h, unstable for others (FE)

3) Bad methods – unstable for any value of h (explicit mid point)

Now formalize these ideas

Test problem $\dot{x} = \lambda x$ $x(0) = 1$

$x(t) = e^{\lambda t}$ λ complex

Simple problem to study the behavior

General multistep method:

$$\sum_{i=0}^{p} \left[\alpha_i x_{n-i} + h \beta_i \dot{x}_{n-i}\right] = 0$$

$$\dot{x}_{n-i} = \lambda x_{n-i}$$

$$\sum_{i=0}^{p} \left[\alpha_i x_{n-i} + \underbrace{h\lambda}_{=\sigma} \beta_i x_{n-i}\right] = 0$$

$$\sum_{i=0}^{p} (\alpha_i + \sigma \beta_i) x_{n-i} = 0$$

The integration method is finding the solution of this difference equation

Try a solution of the form $x_n = c r^n$

$$(\alpha_0 + \sigma \beta_0) c r^n + (\alpha_1 + \sigma \beta_1) c r^{n-1} + \cdots + (\alpha_p + \sigma \beta_p) c r^{n-p} = 0$$

$$(\alpha_0 + \sigma \beta_0) r^p + (\alpha_1 + \sigma \beta_1) r^{p-1} + \cdots + (\alpha_p + \sigma \beta_p) = 0$$

$\Rightarrow$ r is a root of the following polynomial

$$(\alpha_0 + \sigma \beta_0) z^p + (\alpha_1 + \sigma \beta_1) z^{p-1} + \cdots + (\alpha_p + \sigma \beta_p)$$

The polynomial will have p roots (complex)

Distinct roots: $r_1, r_2, \ldots r_p$

$$\longrightarrow x_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_p r_p^n$$

Multiple roots: r_i is of multiplicity m_i

$$\longrightarrow x_n = \left(c_{i0} + c_{i1} n + c_{i2} n^2 + \cdots c_{im_i} n^{m_i - 1}\right) r_i^n$$

Thus x_n is stable under the following conditions:

1) $|r_i| < 1$ for all i

2) $|r_i| = 1$ and $m_i = 1$, other $|r_j| < 1$

Unstable when

1) $|r_i| > 1$ for some i

2) $|r_i| = 1$ and $m_i > 1$

Examples: $\dot{x} = \lambda x$

1) Explicit mid point method

$$x_n = x_{n-2} + 2h \underbrace{\dot{x}_{n-1}}_{\lambda x_{n-1}}$$

$$x_n - 2\overbrace{h\lambda}^{\sigma} x_{n-1} - x_{n-2} = 0$$

The associated polynomial is

$$z^2 - 2\sigma z - 1$$

The roots of which are $z_{1,2} = -\sigma \pm \sqrt{\sigma^2 + 1}$

one root $|z| > 1$

i.e. the method is unstable

BE $\quad x_n = x_{n-1} + h \underbrace{\dot{x}_n}_{\lambda x_n}$

$$x_n - \overbrace{\lambda h}^{\sigma} x_n - x_{n-1} = 0$$

$$(1-\sigma) x_n - x_{n-1} = 0$$

The associated polynomial is $(1-\sigma)z - 1$

roots are $z = \frac{1}{1-\sigma}$

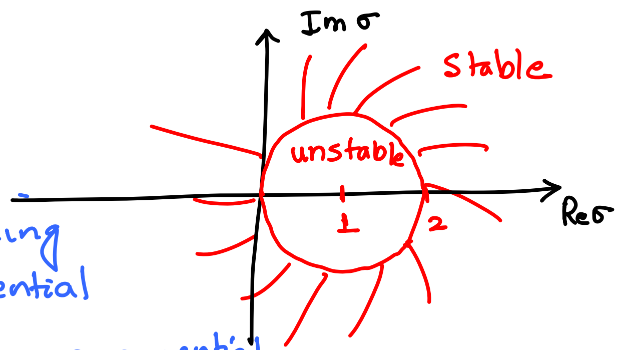
$|z| \leq 1 \Rightarrow \frac{1}{|1-\sigma|} \leq 1$ or $|1-\sigma| \geq 1$

Recall $\sigma = u + jv$ is a complex number

$$|1-u-jv| = (u-1)^2+v^2 \geq 1^2$$

$\dot{x}(t) = \lambda x$

$x(t) = e^{\lambda t}$



For $Re(\lambda) < 0$ decaying exponential

For $Re(\lambda) > 0$ growing exponential

but BE will damp the solution

Do not use BE for oscillators

Region of absolute stability

For a LMS method this is the set of $\sigma = \lambda h$ (complex) such that all solutions of the difference equation

$$\sum_{i=0}^{p} (\alpha_i + \sigma \beta_i)\, x_{n-i} = 0$$

remain bounded as $n \to \infty$

if and only if the roots of the associated polynomial

$$\sum_{i=0}^{p} (\alpha_i + \sigma \beta_i)\, z^{p-i}$$

are inside the complex unit circle $|z| \leq 1$ and the roots for which $|z| = 1$ are of multiplicity 1

Stable method

if all solutions of difference equation obtained by setting $\sigma = 0$ are bounded as $n \to \infty$

i.e. the region of absolute stability includes $\sigma = 0$

PNJLIM Usage

```
double pnjlim();

if (iteration_count == 0) {
        Vd = VD_INIT;
        inst->vold = VD_INIT;
} else {
        Vd = Xk[na] - Xk[nb];
}
Vold = (inst->vold);

// Limit voltage
Vd =pnjlim(Vd, Vold, Vt, vcrit, &icheck);

inst->vold = Vd;                    // save old voltage
```

LIMVDS/FETLIM Usage

```
double limvds(), fetlim();

if (iteration_counter == 0) {
        Vgs = VGS_INIT; Vds = VDS_INIT;
        …
        inst->vgs_old = ABS(Vgs);
        inst->vds_old = ABS(Vds);
} else {
        Vgs = Xk[g]-Xk[s];
        Vds = Xk[d]-Xk[s];
         …
}
Vgs = type*Vgs;
Vds = type*Vds;
…
Vgs_old = inst->vgs_old;
Vds_old = inst->vds_old;
Vgs = fetlim(Vgs,Vgs_old,Vto);
Vds = limvds(Vds,Vds_old);
inst->vgs_old = Vgs;
inst->vds_old = Vds;
```

Linear Multistep Methods – Stability

$$\sum_{i=0}^{p} \alpha_i x_{n-i} + h\beta_i \dot{x}_{n-i} = 0; \text{ Testproblem } \frac{d}{dt}x(t) = \lambda x(t), x(0) = 1 \qquad (1)$$

$$\sum_{i=0}^{p} (\alpha_i + \beta_i h\lambda) x_{n-i} = \sum_{i=0}^{p} (\alpha_i + \sigma\beta_i) x_{n-i} = 0 \qquad (2)$$

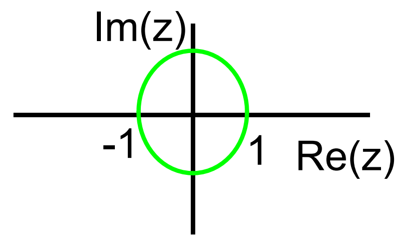
- A method is stable if all solutions of the associated difference equation (2) obtained by setting σ=0 remain bounded as $n\to\infty$
- The region of absolute stability of a method is the set of σ (complex) such that all solutions of (2) remain bounded as $n\to\infty$
- Note: A method is stable if its region of absolute stability contains the origin, i.e., σ=0

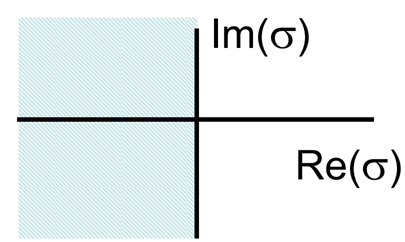
Stability

The region of absolute stability of a method is the set of σ such that all the roots of $\sum_{i=0}^{p} (\alpha_i + \sigma\beta_i) z^{p-i} = 0$ are inside or on the complex unit circle, i.e., $|z| \le 1$, and the roots for which $|z| = 1$ are of multiplicity 1

A method is A-stable if the region of absolute stability contains the entire left-half plane (Re(σ)<0)

TR is an A-stable method





From: A. Nardi

Stability

- **Each method is associated with two polynomials of coefficients α and β:**
  - α: associated with past function values (x_{n-i})
  - β: associated with past derivative values (x'_{n-i})
- **Stability: roots of α polynomial must satisfy $|z|\leq 1$ and be of multiplicity 1 for $|z|=1$**
- **Absolute stability: roots of $(\alpha+\sigma\beta)$ polynomial must satisfy $|z|\leq 1$ and be of multiplicity 1 for $|z|=1$**

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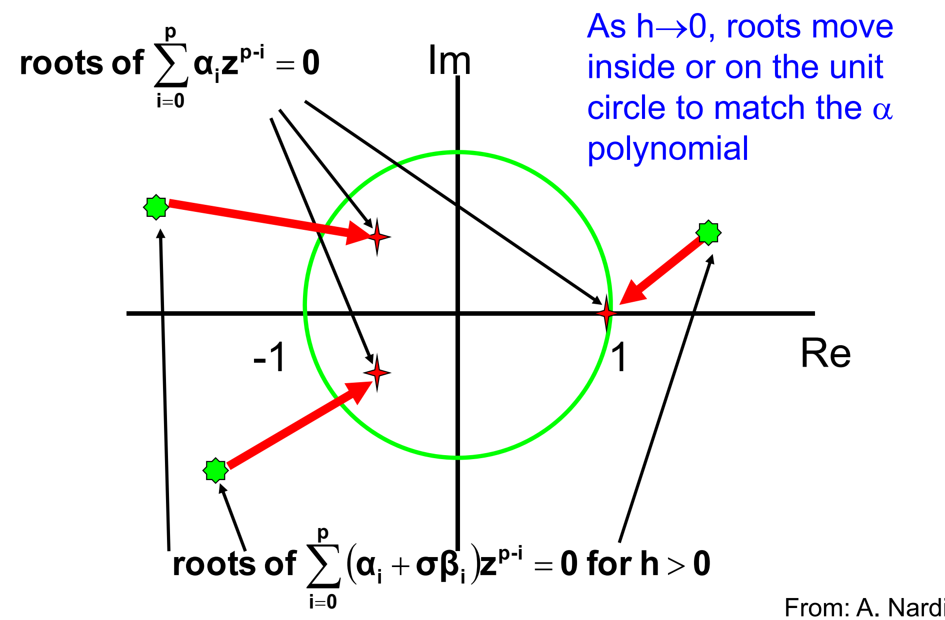
Stability and Region of Absolute Stability



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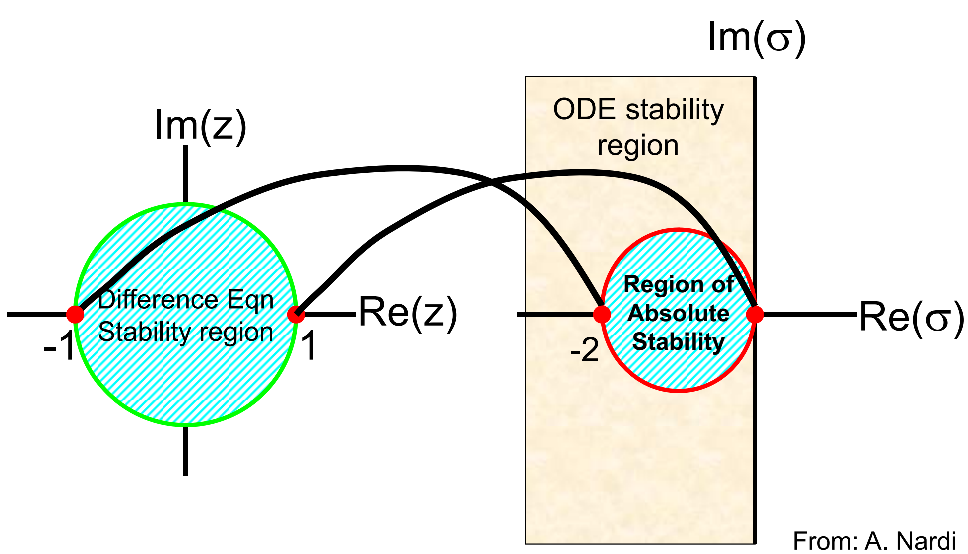
FE Region of Absolute Stability

Forward Euler $\mathbf{z = 1 + \sigma}$



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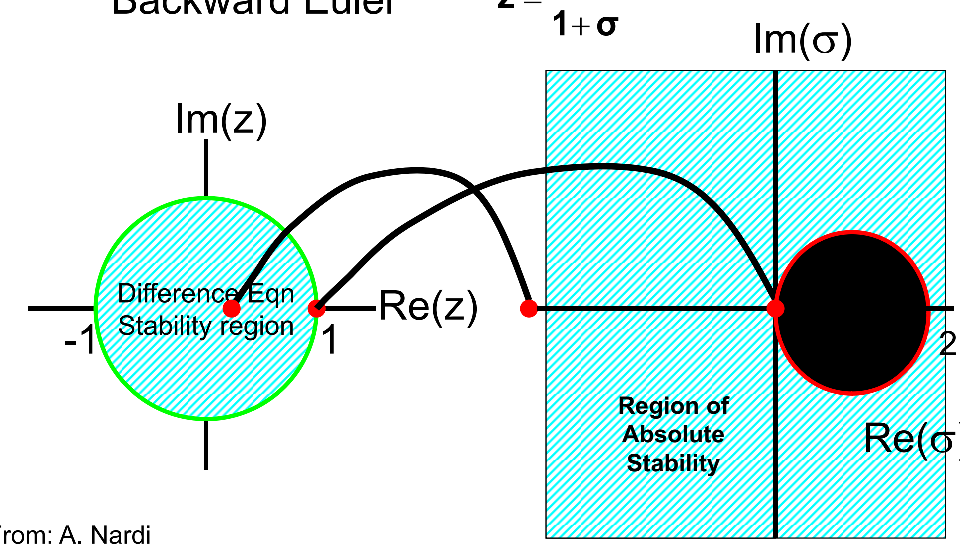
BE Region of Absolute Stability

Backward Euler $\mathbf{z = \dfrac{1}{1+\sigma}}$



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