

A Preface to a "Physics-Style" Analysis of the Infinite Length T-Line

The following pages derive the two essential defining characteristics of a transmission line; $Z_0 + V_p$.

With $Z_0 + V_p$ we can determine per unit length inductance or capacitance, or the delay of the line given its length.

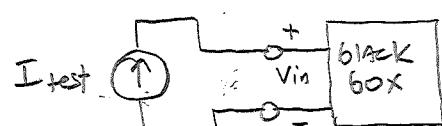
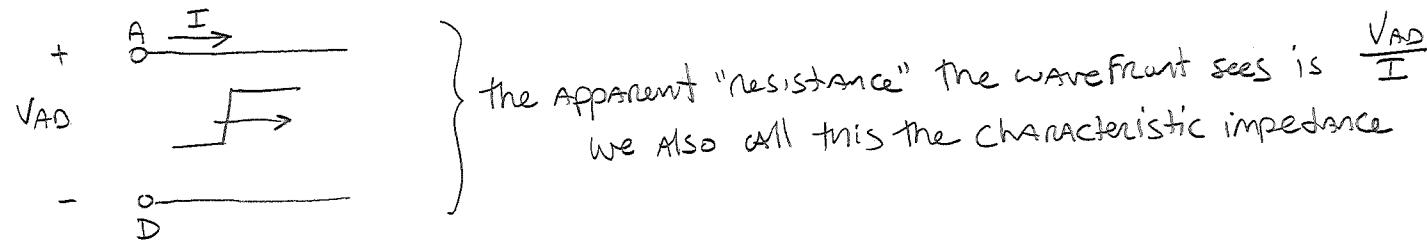
The analysis begins by finding the current into and the voltage across the input to a T-line. This yields two equations:

$$V = \mathcal{L} I_{up} + I = C \mathcal{E}_p V ; \text{ where } \mathcal{L} + C \text{ are the per unit length inductance + capacitance}$$

These two equations are then manipulated to find both $V_p + Z_0$

V_p is found by substitution.

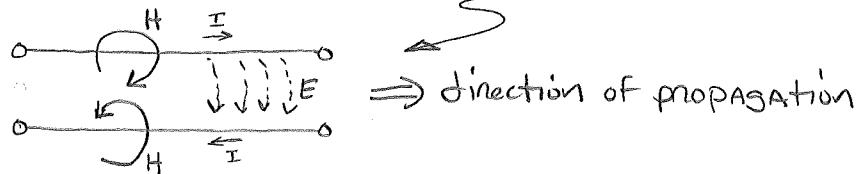
Z_0 is found by dividing V by I (ohms law)



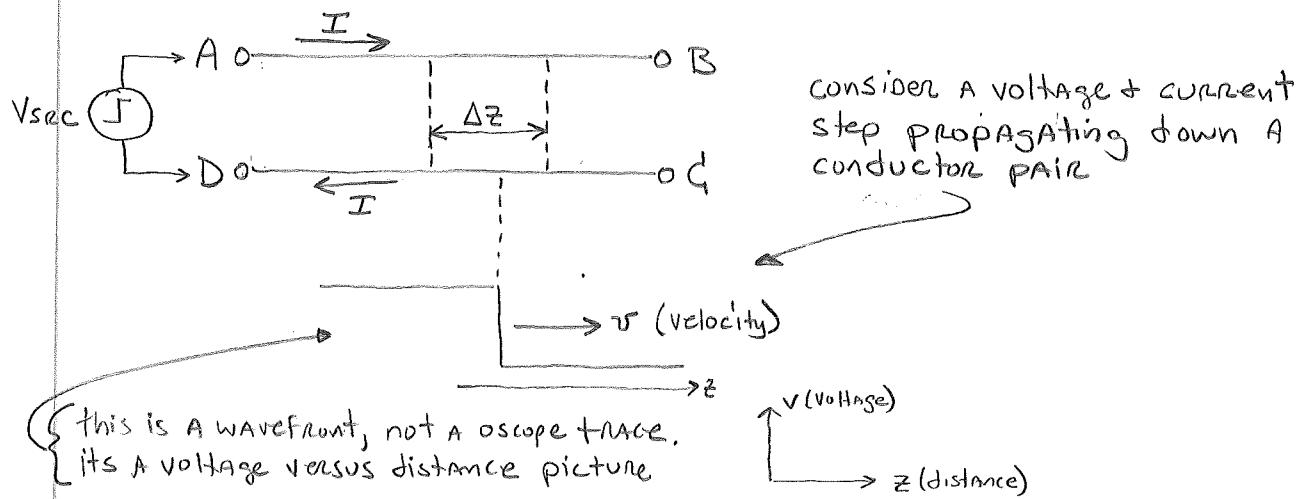
R_{in} of black box is $\frac{V_{in}}{I_{test}}$

A "Physics-style" Analysis of the Infinite Length T-line (lossless)

- A physics approach
- deriving Z_0 & velocity of propagation v
- Assuming TEM (transverse electromagnetic mode) propagation



- in TEM propagation, H & E fields have no components in the direction of propagation



Important points to notice:

- there is no change in current or voltage except right at the edge of the wavefront.
- At every point to the left of the step, ...
 - the voltage between the two wires is V_{AD}
 - the current along AB is I
 - the current along DC is I

Magnetic Field Flux Φ is defined as:

$$\Phi = \int_{(\text{Phi})} \mathbf{B} \cdot d\mathbf{s}; \quad \text{where; } \mathbf{B} = \text{magnetic field vector}$$

$d\mathbf{s}$ = incremental area vector
 \cdot = vector dot product

- the integral is taken over the surface for which Φ is defined

Inductance of Any device is defined as a proportionality constant between the magnetic flux it holds and the current it carries, thus,

$$L = \frac{\Phi}{I}.$$

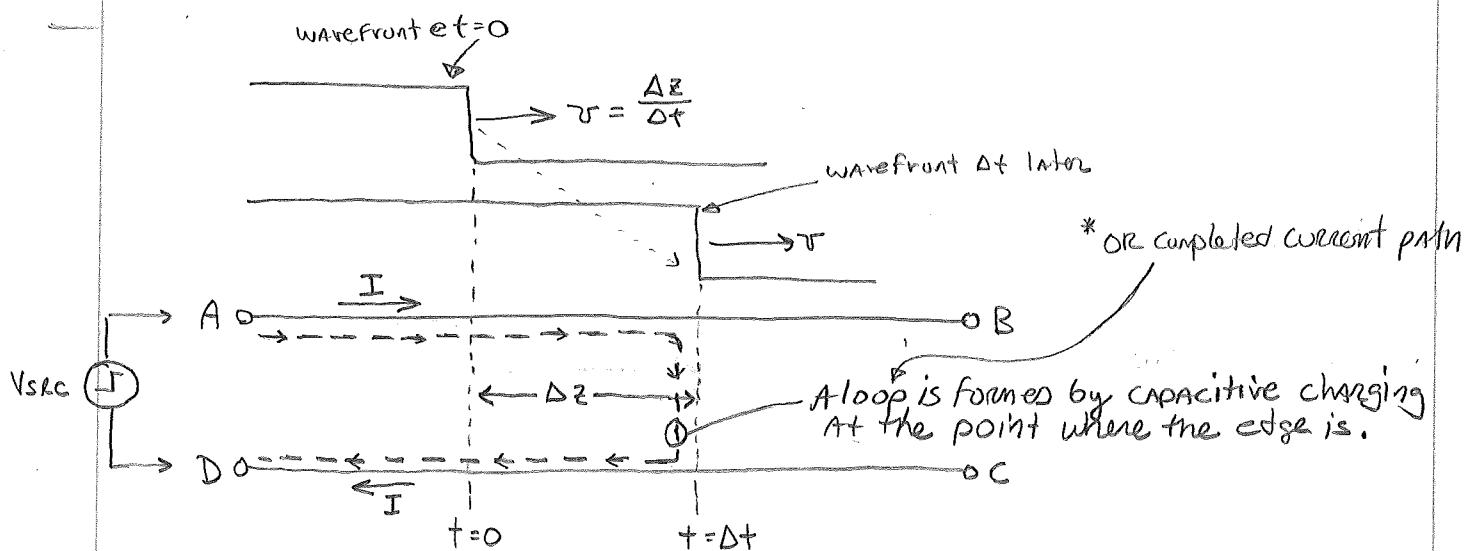
And

$$\Phi = L \cdot I$$

{ more current through the electromagnet makes for a stronger magnet because Φ is increased.

In a transmission line we can say,

$$L = \frac{\phi}{I}; \quad \text{where } L \text{ & } \phi \text{ are inductance & flux measures per unit length}$$



As the wave progresses, the magnetic flux changes. Why?
From before:

$$\Phi = LI \text{ thus } \Delta \Phi = \Delta(LI)$$

However, in the T-line, L is measure per unit length & I is constant as the wave progresses. So,

$$\Delta \Phi = \Delta L(I)$$

And since $L = \frac{L}{z}$, $L = Lz + \Delta L = Lz + \Delta L$

$$\text{so, } \Delta \Phi = L \Delta z (I)$$

$$\text{Faraday's Law says: } \text{EMF} = -\frac{d\Phi}{dt}$$

This is the "back EMF" generated when the voltage/current step progresses down the line.

KVL Around the loop ABCD gives us:

$$\text{KVL}_{ABCD}: V_{AD} - \frac{\Delta \Phi}{\Delta t} = 0$$

$$V_{AD} = \frac{\Delta \Phi}{\Delta t} ; \text{ And from before, } \Delta \Phi = L \Delta z(I), \text{ so}$$

$$V_{AD} = \frac{L \Delta z(I)}{\Delta t}$$

$$V_{AD} = L I \frac{\Delta z}{\Delta t} \quad \text{Velocity of the wavefront}$$

$$\textcircled{1} \quad V_{AD} = L I v$$

Rewriting this as $Iv = \frac{V_{AD}}{L}$ we see that the velocity and magnitude of the current step is limited by the line inductance. ($I = \frac{V_{AD}}{L v}, v = \frac{V_{AD}}{L I}$)

Now, let's take a look at the effects of the electric field (capacitance). T-line behavior is a result of the interaction of the L + C elements. Like inductance, capacitance of a device is a proportionality constant between electric charge and voltage between the terminals of the device.

$$C = \frac{Q}{V} \quad \begin{matrix} \text{(charge)} \\ \text{(voltage)} \end{matrix} \quad V_c = \frac{1}{C} \quad V = \frac{Q}{C} \quad \left\{ \begin{matrix} \text{more charge, higher} \\ \text{voltage} \end{matrix} \right.$$

So now we can calculate the magnitude of the current step moving down the line.

$$I = \frac{\Delta Q}{\Delta t} ; \text{ And since } Q = CV \text{ from above}$$

$$I = \frac{\Delta (CV_{AD})}{\Delta t} ; V_{AD} \text{ is considered constant}$$

$$I = \frac{\Delta C}{\Delta t} V_{AD}$$

In our T-line we can say

$$C = \frac{L}{Z} \quad (\text{capacitance is expressed in per unit length})$$

thus,

$$C = Cz \quad \text{and} \quad \Delta C = C \Delta z$$

from before, we had :

$$I = \frac{\Delta C}{\Delta t} V_{AD} ; \text{ now rewrite as}$$

$$I = \frac{C \Delta z}{\Delta t} V_{AD} \quad \text{velocity of wavefront}$$

$$(2) \quad I = C_v V_{AD}$$

Given the current into the line this tells us that the speed & magnitude of the voltage wave are limited by the line capacitance.

$$V_{AD} v = \frac{I}{C} \Rightarrow V_{AD} = \frac{I}{C_v} \quad \text{or} \quad v = \frac{I}{C V_{AD}} \quad \left\{ C \text{ remains in the denominator} \right.$$

To solve for velocity v , substitute (2) into (1)

$$(1) \quad V_{AD} = Z I v$$

$$(2) \quad I = C_v V_{AD}$$

$$V_{AD} = Z C_v V_{AD} v$$

$$1 = Z C v^2$$

$$v^2 = \frac{1}{Z C}$$

$$v = \sqrt{\frac{1}{Z C}}$$

$$v_p = \frac{1}{\sqrt{Z C}}$$

$\left\{ \begin{array}{l} \text{this is the velocity of propagation of the} \\ \text{wavefront} \\ Z + C \text{ are inductance + capacitance per} \\ \text{unit length} \end{array} \right.$

Now, let's find the equivalent resistance the wavefront experiences as it propagates down the line. This is also called the characteristic impedance Z_0 .

To find "resistance" we want to find $R = \frac{V}{I}$ so we divide eqn (1) by (2).

eqn (1)

eqn (2)

$$V_{AD} = I L v$$

$$I = V_{AD} C v$$

$$\frac{V_{AD}}{I} = \frac{I L v}{V_{AD} C v}$$

$$V_{AD}^2 C v = I^2 L v ; \text{ cross mult.}$$

$$\frac{V_{AD}^2}{I^2} = \frac{L}{C} ; \text{ take sqrt, both sides}$$

"resistance"

$$\Rightarrow \boxed{\frac{V_{AD}}{I}} = \sqrt{\frac{L}{C}}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

Z_0 is the characteristic impedance of the line. It is the effective resistance the wavefront experiences as it travels down the T-line.

A uniform lossless transmission line can be completely defined by its characteristic impedance Z_0 & its velocity of propagation v_p .

Summary of T-Line Parameters:

$$Z_0 = \sqrt{\frac{L}{C}} \quad v_p = \frac{1}{\sqrt{L C}} \quad t_{delay} = L \sqrt{\frac{L}{C}}$$

$$L = \frac{Z_0}{v_p} \quad C = \frac{1}{Z_0 v_p}$$

or \sqrt{LC} : when L & C are the total inductance and capacitance of the entire length of T-line

Maxwell's equations also tell us that the velocity of propagation of a lossless T-line can be derived from the material properties of the medium.

$$v_p = \frac{1}{\sqrt{\epsilon_r \mu_r}} = \frac{c}{\sqrt{\epsilon_r \mu_r}} \quad \text{where;}$$

ϵ = dielectric constant - describes how an electric field influences the organization of charges in a medium, as well as the opposition encountered when forming an E field in a medium. $\propto \epsilon_0$

ϵ_r = relative permittivity - $\epsilon_r = \frac{\epsilon}{\epsilon_0}$; where ϵ_0 is the permittivity of a vacuum (1) in F/m. High ϵ_r supports a greater electric field.

μ = magnetic permeability - describes how a magnetic field influences the magnetic dipoles in a material. It is a measure of the ability of a material to support the formation of a magnetic field within itself. $\propto \mu_0$

μ_r = relative permeability - $\mu_r = \mu/\mu_0$; where $\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$

Since we will be concerned with strictly non-magnetic T-lines,

$$\underline{v_p = \frac{c}{\sqrt{\epsilon_r}}}$$

Propagation Speeds in different dielectrics

Material	ϵ_r	$v_p (\text{mm/ns})$	$\frac{300 \text{ mm}}{\text{ns}}$
Alumina (Al_2O_3)	9.5	97	
FR-4	4.5	141	
SiO_2	3.9	151	
Rogers 6002	3.0	173	
Polyethylene	2.3	197	
Teflon	2.0	212	
Air	1.0	300	

FR-4

Costs for PCBs can be as low as \$1.50 / ft² to \$100,000 / ft² for high-performance laminates such as made by Rogers Corp.*

* BASE MATERIALS FOR HIGH-SPEED HIGH FREQUENCY BOARDS, Rick Hartley, "Printed Circuit Boards and Assembly", March 2002